

MAJORANA AND THE QUASI-STATIONARY STATES IN NUCLEAR PHYSICS

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ABSTRACT. A complete theoretical model describing artificial disintegration of nuclei by bombardment with α -particles, developed by Majorana as early as in 1930, is discussed in detail alongside the basic experimental evidences that motivated it. By following the quantum dynamics of a state resulting from the superposition of a discrete state with a continuum one, whose interaction is described by a given potential term, Majorana obtained (among the other predictions) the explicit expression for the integrated cross section of the nuclear process, which is the direct measurable quantity of interest in the experiments. Though this is the first application of the concept of quasi-stationary states to a Nuclear Physics problem, it seems also that the unpublished Majorana's work anticipates by several years the related seminal paper by Fano on Atomic Physics.

1. NUCLEAR PHYSICS STUDIES IN ROME

After the advent of Enrico Fermi to the chair of Theoretical Physics at the University of Rome in 1926, a group of young people was started to form around him, mainly due to the decisive support of Orso Mario Corbino, the Director of the Institute of Physics in Rome. Aimed to a powerful relaunch of Physics in Italy, upon Fermi's suggestion Corbino firstly hired Franco Rasetti as his assistant, which was an excellent experimentalist who studied with Fermi in Pisa and later worked with him in Florence. Then, after a famous appeal by Corbino to the students of Engineering, in order to convince the most brilliant ones to register in Physics, Emilio Segrè and Edoardo Amaldi decided to join the rising group of Fermi, though they were still students. After some months, at the end of 1927, Ettore Majorana passed as well to Physics, the details of the first meeting by Majorana and Fermi being well known [1][2][3].

At that time, Fermi was working on the statistical model of atoms (the Thomas-Fermi model), and the activities of his and Rasetti's group were thus devoted to researches on Atomic Physics. However, after having obtained several significant results in atomic and molecular spectroscopy, around 1930 some people in the group realized that such a field could offer no longer any great prospect, and Fermi himself predicted that the interest would have shifted from the study of the external parts of the atom to its nucleus. No general consensus, however, created initially inside the group. Segrè, for example, recalls that [1]:

My personal reaction was that we had just learned spectroscopic techniques, with which we were reaping good results, and that we might persist in that field a little longer. I was, however, open to Fermi's arguments. Amaldi and Rasetti also had their points

of view, and we had long, lively discussions on the subject. As was to be expected, Fermi's ideas prevailed, although everybody was left free to do what he liked best. Thus I continued to work experimentally on spectroscopy until we started on neutron work. However, we all increased our reading on nuclear subjects.

The spectroscopic activities, in fact, continued side by side in the acquisition of theoretical knowledge and experimental technologies required by Nuclear Physics, till 1934, when the Fermi's group discovered the radioactivity induced by neutrons and the important properties of slow neutrons [4]. As a bridge between spectroscopy and Nuclear Physics, we find two theoretical works of 1930 [5] and 1933 [6]. In the first of these, performed by Fermi, the magnetic moments of sodium, rubidium and cesium nuclei were deduced from the hyperfine atomic structures observed in the spectra of these elements. Instead, in the second one, Fermi and Segrè performed quite exhaustive study on the nuclear magnetic moments of 17 element, aimed to show that hyperfine structures of atomic spectra could be completely explained by the nuclear magnetic moment, and that there were no other nuclear forces at play. The work by Fermi and Segrè resulted into some involved calculations, who required the computation of the coupling constants of single electrons and that of the perturbations of the electronic terms, and the authors themselves explicitly acknowledged some help on these by Ettore Majorana. According to direct testimonies [1][2] and, mainly, to what reported in his research notebooks (see, for example, [7]), Majorana actively participated in scientific discussions inside the Fermi's group, but we are left with only one published paper of him on Nuclear Physics topics [8], dealing with the known Majorana-Heisenberg exchange forces in nuclei. Practically nothing is known, until now, of his previous works on these matters; here we will focus just on this. We firstly point out that, indeed, Majorana was the *first* in Rome to study nuclear physics, at least since 1929 when (on July 6) he defended his master thesis on "The quantum theory of radioactive nuclei". However, quite unexpected, he continued to study such topics for several years, also independently from the main researches of the Fermi's group, till his famous theory of nuclear exchange forces of 1933. In the following we will report on relevant studies conducted mainly on some unpublished notes by Majorana ranging on this time period, revealing previously unknown interesting results obtained by him on Nuclear Physics topics. The resulting picture, properly included in its contemporary historical framework, points out once more the extraordinary abilities of the Italian physicist.

2. EXPLORING THE NUCLEUS' SECRETS WITH α PARTICLES

The first steps of Nuclear Physics were moved in the realm of radioactive phenomena, implying the emission of some "rays" from several substances like uranium. It was Rutherford to realize that such rays were made of two different components, the first one, easily absorbed by matter, being denoted as α radiation, while the second one, much more penetrating, as β radiation,. Radioactive phenomena involving α emission were readily recognized to induce a profound modification of the atom (or, as was realized later, of the nucleus), due to the large mass of the emitted α particles. The extremely high values of the energies released in these processes, furthermore, led Rutherford himself in 1904 to postulate that the causes of radioactivity should be searched inside the atom and later, in 1911, to interpret the famous experiment of Geiger and Marsden, on the α bombardment of atoms, in

terms of his nuclear atomic model. Contextually, artificial radioactivity induced by particle irradiation was discovered and, in particular, the emergence of protons as disintegration products of nitrogen bombarded with α particles, led to the conclusion that the masses of nuclei were primary due also to protons, besides α particles. The problem of the possible presence of electrons inside the nucleus, driven by the fact that such light particles were emitted in β -decays, remained instead controversial and unsolved until the discovery of the neutron by Chadwick in 1932 and the formulation of the Fermi theory of β -decay one year later (a detailed account of this question may be found in [9]). Here we only quote, in passing, a far-seeing intuition of Majorana on this problem in 1929:

It seems to me that the problem of the aggregation of protons and electrons in nuclei cannot have solutions, though approximate, until the problem of the constitution of protons and electrons themselves has been solved. This is for a very simple reason: the size of complex nuclei... are of the same order of that of electrons (obviously, this is evaluated on classical grounds. Quantum mechanics has not brought, and *cannot brought by itself*, any light on this question, since no relation between e, h, m can hold due to dimensional reasons). Although such statements are largely vague, we can expect that something is hidden in them. [10]

Much information on nuclear structure and dynamics came out, in 1920's and 1930's, from experiments on nuclei bombarded with α particles (as a later example, we mention the discovery of the neutron anticipated by researches on α irradiation on boron and berillium), as one could obviously expect due to the peculiar transformations induced by such massive particles. On the theoretical side, the first important achievement was due to Gamow, that provided the theory for the spontaneous α emission from a nucleus in the framework of Quantum Mechanics [11]. Just after the appearance of the Gamow paper, in 1928 Majorana, who was studying Nuclear Physic (as well as some Atomic Physics topics at the same time) in preparing his master thesis supervised by Fermi, considered also the inverse process of absorption of an α particle by a radioactive nucleus [12]. It is quite instructive to see how Majorana approached such a problem:

Let us consider the emission of an α particle by a radioactive nucleus and assume that such a particle is described by a quasi-stationary wave. As Gamow has shown, after some time this wave scatters at infinity. In other words, the particle spends some time near the nucleus but eventually ends up far from it. We now begin to study the features of such a quasi-stationary wave, and then address the inverse of the problem studied by Gamow. Namely, we went to determine the probability that an α particle, colliding with a nucleus that has just undergone on α radioactive transmutation, will be captured by the nucleus of the element preceding the original one in the radioactive genealogy.

After lengthy calculations (the complete paper may be found, in English, in [12]), Majorana indeed succeeded in obtaining the expression for the absorption probability (in two different ways: by using mechanical or thermodynamical arguments),

$$\frac{n}{N} = \frac{2\pi^2 \hbar^3}{m^2 v T} \quad (1)$$

(T being the life-time and m, v the mass and velocity of the α particle, respectively), by relating it to nuclear α -decay quantities and showing that such probability is “completely independent of any hypothesis on the form of the potential near the nucleus, and that it only depends on the lifetime T .” Intriguing results were then obtained in 1929 from the artificial disintegrations of nuclei by means of α -particles. It was already known that, when bombarding certain elements by α -particles, protons were emitted as a result of the disintegration of the considered nuclei. On the basis of Blackett’s photographs [13] of the disintegration of the nitrogen nucleus, it was assumed that in such a process the α -particle is absorbed by the atomic nucleus, which is thereby transformed into a nucleus of the element of the next higher atomic number. According to this view, by applying energy-momentum conservation laws, simple relations between the energy of the incident α particle and that of the emitted proton should hold and, furthermore, the change of energy from the original to the final nucleus should be a fixed amount. In fact, if we denote with $-E_p^0$ an energy level of a proton in a given nucleus and with $-E_\alpha^0$ that for an α -particle, when such a particle with energy E_α is captured by the nucleus, the energy of the proton emitted in the disintegration will be $E_p = E_\alpha + E_\alpha^0 - E_p^0$ (neglecting the small kinetic energy of the residual nucleus). Thus, in general, protons with definite energy will be emitted during this process, so that their energy spectrum is a discrete one. However, Rutherford and Chadwick in 1929 [14] clearly showed that, at least in some cases (for example in the transition between aluminium and silicon), this is not true, and the change of energy was not always the same. This should lead, as a consequence, to assume that the energy levels of the protons in the α particles in the nucleus are not well defined, in contrast to what emerged from the application of Quantum Mechanics to the nuclear structure. Such evidences were later confirmed by Chadwick, Constable and Pollard [13] but, in the meantime, a novel idea for interpreting them came out. Chadwick and Gamow [15], in fact assumed that, in some cases, the disintegration of the nucleus may occur by the ejection of a proton *without* capture of the α -particle. In this occurrence, the energy of the incident α -particle is distributed between the emitted proton and the escaping α -particle (neglecting the recoiling nucleus), so that the disintegration protons may have any energy between 0 and $E_\alpha - E_p^0$, thus explaining the experimental evidences. As a consequence, if both types of disintegration (with and without α -capture) take place, in the simplest case the energy spectrum of the emitted protons is composed of a continuous spectrum and, after its endpoint, a line. The importance of the identification of the continuous spectrum was readily recognized by Chadwick, Constable and Pollard themselves: “It is most important to find the continuous spectrum of protons corresponding to disintegration without capture, for this gives immediately the level of the proton in the nucleus.” [13]. The theoretical interpretation of the results from (α, p) reactions was mainly due to Chadwick and Gamow but, to the best of our knowledge, no dynamical theory for those processes describing the superposition of a continuous spectrum and a discrete level was ever published. Recently, however, we have realized that such a theory was effectively elaborated by Majorana in 1930, although he never reported it in any journal, being contained in his personal notebooks. In the following we will give a detailed account of the Majorana theory (see [16]), pointing out the interesting, unknown results obtained by the Italian physicist.

3. MAJORANA GENERAL THEORY FOR QUASI-STATIONARY STATES

3.1. First problem. Let us consider, in general, a physical system for which a discrete state ψ_0 exists with energy E_0 together with a continuum one ψ_W with energy $E_0 + W$ (the state ψ_W is normalized with respect to dW). The perturbation linking the discrete state with the continuum ones is denoted by Majorana as:

$$I_W = \int \bar{\psi}_0 H_p \psi_W d\tau, \quad (2)$$

where H_p is the perturbation potential and $d\tau$ is the volume element. The problem is that of obtaining the perturbed eigenfunctions ψ'_W of the total hamiltonian H :

$$H\psi'_W = (E + W)\psi'_W, \quad (3)$$

with

$$H\psi_0 = E_0\psi_0 + \int \bar{I}_W \psi_W dW, \quad (4)$$

$$H\psi_W = (E_0 + W)\psi_W + I_W\psi_0. \quad (5)$$

Majorana finds the analytic solutions for the eigenfunctions, which are expressed as follows:

$$\psi'_W = \frac{1}{N_W} \left\{ \psi_0 - \int \bar{I}_{W'} \frac{\psi_{W'}}{W' - W} dW' + a\psi_W \right\} \quad (6)$$

where $N_W = \sqrt{|a|^2 + |b|^2}$ is a normalization factor written in terms of the parameters:

$$a = I_W^{-1} \left\{ W + \int |I_W|^2 \frac{dW'}{W' - W} \right\}, \quad (7)$$

$$b = \pi I_W \quad (8)$$

(here and below, the integrals are assumed to be evaluated at their principal value). In such a way, the discrete state ψ_0 may be expanded in terms of the perturbed eigenfunctions ψ'_W as:

$$\psi_0 = \int \frac{1}{N_W} \psi'_W dW. \quad (9)$$

In order to get explicit results, Majorana then considers the approximate case where terms higher than the second in I_W are neglected, so that the quantities:

$$I_W \simeq I, \quad \int |I_W|^2 \frac{dW'}{W' - W} \simeq k \quad (10)$$

may be regarded as constants. By setting $W = \varepsilon - k$ we have $a = \varepsilon/I$, $b = \pi I$, $N = \sqrt{\varepsilon^2/I^2 + \pi^2|I|^2}$ and the perturbed eigenfunctions are:

$$\psi'_W \simeq \frac{1}{\sqrt{\varepsilon^2/I^2 + \pi^2|I|^2}} \left\{ \psi_0 - \bar{I} \int \frac{\psi_{W'}}{W' - W} dW' + \frac{\varepsilon}{I} \psi_W \right\}. \quad (11)$$

Now, let us assume that at time $t = 0$ the system is in the state ψ_0 . If the time-dependent factor of ψ'_W is taken to be:

$$e^{-iEt/\hbar} = e^{-i(E_0 - k)t/\hbar} e^{-i\varepsilon t/\hbar}, \quad (12)$$

the time evolution of the state of the system is given by:

$$\psi = e^{-i(E_0 - k)t/\hbar} \left\{ e^{-t/2T} \psi_0 + \int \frac{\bar{I}}{\varepsilon + i\pi|I|^2} \left(e^{-i\varepsilon t/\hbar} - e^{-t/2T} \right) \psi_W d\varepsilon \right\}. \quad (13)$$

Here,

$$\frac{1}{T} = \frac{2\pi}{\hbar} |I|^2 \quad (14)$$

gives the transition probabilities per unit time between the initial (discrete) state ψ_0 and the states ψ_W .

3.2. Second problem. The next problem considered by Majorana was to study the transitions between one discrete state ψ_0 with energy E_0 and two continuum ones ψ_W and ϕ_W , both with energy $E_0 + W$. In a way analogous to the previous one, the two perturbations are written as:

$$I_W = \int \bar{\psi}_0 H_p \psi_W dW, \quad (15)$$

$$L_W = \int \bar{\psi}_0 H_p \phi_W dW, \quad (16)$$

and

$$H\psi_0 = E_0\psi_0 + \int \bar{I}_W \psi_W dW + \int \bar{L}_W \phi_W dW, \quad (17)$$

$$H\psi_W = (E_0 + W)\psi_W + I_W\psi_0, \quad (18)$$

$$H\phi_W = (E_0 + W)\phi_W + L_W\psi_0. \quad (19)$$

Now, for any value of W , we have two stationary states Z_W^1 and Z_W^2 ,

$$HZ_W^1 = (E_0 + W)Z_W^1 \quad (20)$$

$$HZ_W^2 = (E_0 + W)Z_W^2, \quad (21)$$

that can be chosen to be orthogonal between them and normalized in such a way that:

$$Z_W^1 = \frac{1}{N'_W} \left\{ \psi_0 + \varepsilon_W \frac{\bar{I}_W}{Q_W^2} \psi_W + \varepsilon_W \frac{\bar{L}_W}{Q_W^2} \phi_W - \int \frac{\bar{I}_{W'} \psi_{W'}}{W' - W} dW' - \int \frac{\bar{L}_{W'} \phi_{W'}}{W' - W} dW' \right\}, \quad (22)$$

$$Z_W^2 = \frac{L_W \psi_W}{Q_W} - \frac{I_W \phi_W}{Q_W}. \quad (23)$$

where:

$$\varepsilon_W = W + \int |I_{W'}|^2 \frac{dW'}{W' - W} + \int |L_{W'}|^2 \frac{dW'}{W' - W} = W + K_W \quad (24)$$

$$Q_W = \sqrt{|I_W|^2 + |L_W|^2}, \quad (25)$$

$$N'_W = \sqrt{\frac{\varepsilon_W^2}{Q_W^2} + \pi Q_W^2}. \quad (26)$$

Since Z_W^2 are orthogonal to ψ_0 , this can be expanded by using only the states Z_W^1 :

$$\psi_0 = \int \frac{Z_W^1}{N'_W} dW. \quad (27)$$

By using analogous approximations to that introduced above, Majorana then finds the time evolution of the state of the system which is initially in the state ψ_0 :

$$\begin{aligned}\psi &= e^{-i(E_0-k)t/\hbar} e^{-t/2T} \psi_0 \\ &+ \bar{I} \int \frac{\psi_W e^{-iEt/\hbar}}{W + K + i\pi(|I|^2 + |L|^2)} \left(1 - e^{i(W+K)t/\hbar} e^{-t/2T}\right) dW + \\ &+ \bar{L} \int \frac{\phi_W e^{-iEt/\hbar}}{W + K + i\pi(|I|^2 + |L|^2)} \left(1 - e^{i(W+K)t/\hbar} e^{-t/2T}\right) dW, \quad (28)\end{aligned}$$

where

$$\frac{1}{T} = \frac{2\pi}{\hbar} (|I|^2 + |L|^2). \quad (29)$$

The transition probability per unit time between ψ_0 and the states ψ_W or ϕ_W is $2\pi|I|^2/\hbar$ or $2\pi|L|^2/\hbar$, respectively.

3.3. Third problem. Now let us suppose that the system is initially in the continuum state ψ_W and consider the relative probability that at time t the system is in the state ψ_0 , or in the states ψ_W , or in the states ψ'_W with $W' \neq W$. First of all, we note that Majorana “prefers” to use the concept of “number of systems” in a given state rather than that of the relative probability for the system to be in that state. Although the initial state ψ_W is not a stationary state and represents an infinite number of systems, only a finite of them has an energy that differs from $E_0 + W$ by a finite amount, so that we can effectively expect only transitions to states next to ψ_W and ϕ_W . In such framework, the initial state may be expanded in terms of the stationary states Z_W^1 and Z_W^2 , and its time evolution at time t is written (by using the usual above approximation) as:

$$\begin{aligned}\psi &= \frac{1}{N'_W} \frac{\varepsilon I}{|I|^2 + |L|^2} e^{-iEt/\hbar} Z_W^1 + I \int e^{-iE't/\hbar} \frac{Z_{W'}^1}{N'_{W'}(W' - W)} \\ &+ \frac{\bar{L}}{|I|^2 + |L|^2} e^{-iEt/\hbar} Z_W^2 \quad (30)\end{aligned}$$

($E = E_0 + W$, $E' = E_0 + W'$). Here Majorana also notes that, in order to regularize the integral in (30), it is convenient (above and in the following) to use the replacement:

$$\frac{1}{W' - W} \rightarrow \frac{W' - W}{(W' - W)^2 + \alpha^2} \quad (31)$$

and take then the limit $\alpha \rightarrow 0$ at the end of calculations. The expression of ψ in terms of the unperturbed states ψ_0, ψ_W, ϕ_W may be obtained by using the expressions of Z_W^1, Z_W^2 in Eqs. (22) (23). However, Majorana points out that, for later times $t > 0$, it is convenient to express ψ as a sum of two terms ψ_1 and ψ_2 , such that $\psi_1 + \psi_2 = \psi_W$ at $t = 0$ and that ψ_1 substantially describes the phenomenon for sufficiently large times, while ψ_2 is one of the discrete states of the form given in (28). He then finds:

$$\psi = \psi_1 + \psi_2 \quad (32)$$

$$\begin{aligned}
\psi_1 &= e^{-iEt/\hbar} \psi_W + \frac{I}{\varepsilon + i\pi Q^2} e^{-iEt/\hbar} \psi_0 \\
&\quad - \frac{I}{\varepsilon + i\pi Q^2} \int \frac{\bar{I}\psi_{W'} + \bar{L}\phi_{W'}}{\varepsilon' - \varepsilon} e^{-iE't/\hbar} \left(1 - e^{-i(E-E')t/\hbar}\right) dE', \\
\psi_2 &= -\frac{I}{\varepsilon + i\pi Q^2} \left\{ e^{-i(E_0-K)t/\hbar} e^{-t/2T} \psi_0 \right. \\
&\quad \left. + \int \frac{\bar{I}\psi_{W'} + \bar{L}\phi_{W'}}{\varepsilon' + i\pi Q^2} e^{-iE't/\hbar} \left(1 - e^{-i(E'-E_0+K)t/\hbar} e^{-t/2T}\right) dE' \right\}
\end{aligned}$$

($Q = \sqrt{|I|^2 + |L|^2}$, $\varepsilon = E - E_0 + K$, $\varepsilon' = E' - E_0 + K$) with

$$\frac{1}{T} = \frac{2\pi}{\hbar} Q^2. \quad (33)$$

For sufficiently large times, the number of transitions per unit time from ψ_W to the states $\psi_{W'}$ with energy W' close to W is dominated by the resonance term $1/(\varepsilon' - \varepsilon)$ in the expression for ψ_1 . Thus, the number A of transitions to states $\psi_{W'}$ or that B to states $\phi_{W'}$ is estimated as:

$$A = \frac{2\pi}{\hbar} |I|^2 \frac{|I|^2}{\varepsilon^2 + \pi^2 Q^2}, \quad B = \frac{2\pi}{\hbar} |L|^2 \frac{|I|^2}{\varepsilon^2 + \pi^2 Q^2}, \quad (34)$$

respectively.

3.4. Fourth problem. In order to describe several practical applications, Majorana finally introduces some linear combinations of the states ψ_W and ϕ_W , having a definite physical meaning:

$$\psi_W = u_W^1 + u_W^2, \quad \phi_W = v_W^1 + v_W^2, \quad (35)$$

where:

$$u_W^1 = \frac{1}{2} \psi_W - \frac{i}{2\pi} \int \frac{\bar{I}_{W'}}{\bar{I}_W} \frac{\psi_{W'}}{(W' - W)} dW', \quad (36)$$

$$u_W^2 = \frac{1}{2} \psi_W + \frac{i}{2\pi} \int \frac{\bar{I}_{W'}}{\bar{I}_W} \frac{\psi_{W'}}{(W' - W)} dW', \quad (37)$$

$$v_W^1 = \frac{1}{2} \phi_W - \frac{i}{2\pi} \int \frac{\bar{L}_{W'}}{\bar{L}_W} \frac{\phi_{W'}}{(W' - W)} dW', \quad (38)$$

$$v_W^2 = \frac{1}{2} \phi_W + \frac{i}{2\pi} \int \frac{\bar{L}_{W'}}{\bar{L}_W} \frac{\phi_{W'}}{(W' - W)} dW'. \quad (39)$$

The most general stationary state Z_W corresponding to the energy $E_0 + W$ can be written as a linear combination of the states Z_W^1 , Z_W^2 in Eqs (22), (23):

$$Z_W = \lambda Z_W^1 + \mu Z_W^2. \quad (40)$$

By setting

$$Z_W = c\psi_0 + c_1 u_W^1 + c_2 u_W^2 + C_1 v_W^1 + C_2 v_W^2, \quad (41)$$

we have:

$$c = \frac{\lambda}{N'_W}, \quad (42)$$

$$c_1 = \lambda \frac{\bar{I}_W}{N'_W} \left(\frac{\varepsilon_W}{Q_W^2} - i\pi \right) + \mu \frac{L_W}{Q_W}, \quad (43)$$

$$c_2 = \lambda \frac{\bar{I}_W}{N'_W} \left(\frac{\varepsilon_W}{Q_W^2} + i\pi \right) + \mu \frac{L_W}{Q_W}, \quad (44)$$

$$C_1 = \lambda \frac{\bar{I}_W}{N'_W} \left(\frac{\varepsilon_W}{Q_W^2} - i\pi \right) - \mu \frac{L_W}{Q_W}, \quad (45)$$

$$C_2 = \lambda \frac{\bar{I}_W}{N'_W} \left(\frac{\varepsilon_W}{Q_W^2} + i\pi \right) - \mu \frac{L_W}{Q_W}. \quad (46)$$

Majorana then focuses on particular stationary states with $C_2 = 0$; from such condition he determines the coefficients λ, μ and definitely obtains the expressions for c, c_1, c_2, C_1, C_2 . Quantities of particular interest are their squared modulus:

$$|c_1|^2 = \frac{\varepsilon_W^2 + \pi^2(|I_W|^2 - |L_W|^2)^2}{\varepsilon_W^2 + \pi^2 Q^2}, \quad (47)$$

$$|c_2|^2 = \frac{4\pi^2 |I_W|^2 |L_W|^2}{\varepsilon_W^2 + \pi^2 Q^2}, \quad (48)$$

while $|c_2|^2 = 1$ and $|c_1|^2 + |C_1|^2 = 1$. Now, let us consider the usual approximation for which $I_W = I$, $L_W = L$ are constants. In some applications the ratio:

$$p = p(\varepsilon) = \frac{|c_1|^2}{|c_2|^2} \quad (49)$$

has a direct physical meaning, and is here given by

$$p(\varepsilon) = \frac{4\pi^2 |I|^2 |L|^2}{\varepsilon^2 + \pi^2 Q^4}. \quad (50)$$

Its maximum value p_0 (obtained for $\varepsilon = 0$) varies between 0 and 1 (for $|I|^2 = |L|^2$) and, putting $k = |I|^2/|L|^2$, can be expressed as

$$p_0 = \frac{4k}{(k+1)^2} \quad (51)$$

(note that $p_0(k) = p_0(1/k)$). It determines the integral:

$$\int p(\varepsilon) d\varepsilon = \pi^2 Q^2 p_0. \quad (52)$$

In terms of the total disintegration probability of the unstable state ψ_0 ,

$$\frac{1}{T} = \frac{2\pi}{\hbar} Q^2 \quad (53)$$

and the partial disintegration probabilities for transitions to the states ψ_W and ϕ_W ,

$$\frac{1}{T_1} = \frac{2\pi}{\hbar} |I|^2, \quad \frac{1}{T_2} = \frac{2\pi}{\hbar} |L|^2, \quad (54)$$

with

$$\frac{1}{T} = \frac{1}{T_1} + \frac{1}{T_2}, \quad (55)$$

$$\frac{1}{T_1} = \frac{k}{k+1} \frac{1}{T}, \quad \frac{1}{T_2} = \frac{1}{k+1} \frac{1}{T}, \quad (56)$$

we have:

$$\int p(\varepsilon) d\varepsilon = \frac{2\pi\hbar}{T} \frac{p_0}{4} = \frac{2\pi\hbar}{T} \frac{k}{(k+1)^2} = \frac{2\pi\hbar}{T_1} \frac{1}{k+1} = \frac{2\pi\hbar}{T_2} \frac{k}{k+1}. \quad (57)$$

4. APPLICATION TO α -INDUCED NUCLEAR DISINTEGRATION

The general theory outlined above was elaborated by Majorana in order to describe the artificial disintegrations of nuclei by means of α -particles, with and without α -absorption, as recalled earlier. Majorana approaches the problem by considering the simplest case with an unstable state (described by ψ_0) of the system formed by a nucleus plus an α -particle, which spontaneously decays with the emission of an α -particle or a proton. For simplicity, he assumes that such proton or α -particle, coming from the disintegration of ψ_0 , is emitted as an s -wave and that the daughter nucleus is always in its ground state. The initial system formed by the parent nucleus plus the incoming α -particle (in a hyperbolic s -orbit) is described by the states ψ_W , while the final one formed by the daughter nucleus and the free proton (in an s -orbit) is described by the states ψ_W . The state ψ_0 is coupled to both ψ_W and ϕ_W states by the perturbation potential H_p in Eqs. (15), (16). Neglecting the motion of the nucleus, ψ_W represents a converging or diverging flux (number of particles per unit time) of α -particles equal to $1/2\pi\hbar$, while ϕ_W represents an ingoing or outgoing flux of protons equal to $1/2\pi\hbar$. Instead, the non-stationary states u_W^1 or u_W^2 introduced above represent, at large distances, only outgoing or ingoing fluxes of α -particles (their intensity being always $1/2\pi\hbar$), respectively, and similarly for v_W^1 and v_W^2 describing outgoing or ingoing fluxes of protons. The problem at face is that to study the scattering of (a unitary flux per unit area of) α -particles interacting with the parent nucleus and to determine for what number of them the nucleus disintegrates. To this end, Majorana considers a stationary state, representing the incident plane wave plus a diverging spherical wave of α -particles plus a diverging spherical wave of protons, as obtained by a sum of particular solutions. Such solutions are not limited to those corresponding to the parent nucleus plus α -particles with non-vanishing azimuthal quantum numbers, whose form is well known from the theory of Coulomb scattering. They also account for incident α -particles with $\ell = 0$ as well as a diverging wave of α -particles with $\ell = 0$. Moreover, due to the coupling between ψ_0 and ϕ_W , the stationary state must also be composed of an excited (at certain degree) ψ_0 state as well as of a diverging wave of protons. Such a particular solution will then have the form as in Eq. (41) (with $C_2 = 0$) with the coefficients taking the following values, except for a proportionality factor:

$$c = \frac{I_W}{N'_W Q_W}, \quad (58)$$

$$c_1 = \frac{1}{N'_W Q_W} \{ \varepsilon_W - i\pi (|I_W|^2 - |L_W|^2) \}, \quad (59)$$

$$C_1 = -2\pi \frac{I_W \bar{L}_W}{N'_W Q_W}, \quad (60)$$

$$c_2 = \frac{1}{N'_W Q_W} (\varepsilon_W + i\pi Q_W^2). \quad (61)$$

The coefficient c_2 can be determined from the condition that the incoming flux of α -particles is due to the incident plane wave, this incoming flux being equal to $|c_2|^2/2\pi\hbar$. On the other hand, the number of α -particles with $\ell = 0$ impinging on the nucleus per unit time is equal to the flux through a circular cross section, normal to the propagation direction of the wave, with radius $\lambda/2\pi$, λ being the wavelength of the α -particles. Since the incident wave represents a unit flux per unit area, we have:

$$\frac{|c_2|^2}{2\pi\hbar} = \pi \left(\frac{\lambda}{2\pi} \right)^2 = \left(\frac{\lambda^2}{4\pi} \right) = \frac{\pi\hbar^2}{M^2v^2}, \quad (62)$$

M and v being the mass and the velocity of the particles, respectively. From this and from Eqs. (47), (48) the value of $|c_2|^2$ and that of $|c_1|^2$ and $|C_1|^2$ may be obtained:

$$|c_1|^2 = \frac{\hbar\lambda^2}{2} \frac{\{\varepsilon^2 + \pi^2(|I|^2 - |L|^2)^2\}}{\varepsilon^2 + \pi^2Q^4}, \quad (63)$$

$$|C_1|^2 = \frac{\hbar\lambda^2}{2} \frac{4\pi^2|I|^2|L|^2}{\varepsilon^2 + \pi^2Q^4}, \quad (64)$$

$$|c_2|^2 = \frac{\hbar\lambda^2}{2}. \quad (65)$$

The knowledge of the moduli of such coefficients suffices for the problem at face, consisting in the study of only the frequency of disintegration processes, disregarding scattering anomalies that also depend on the phase of c_1 .

The cross section $S(\varepsilon)$ for the disintegration process is given by the outgoing proton flux, $|C_1|^2/2\pi\hbar$:

$$S(\varepsilon) = \frac{\hbar\lambda^2}{4\pi} \frac{4\pi^2|I|^2|L|^2}{\varepsilon^2 + \pi^2Q^4} = \frac{\hbar\lambda^2}{4\pi} p(\varepsilon) \quad (66)$$

(by assuming, at first approximation, that λ, I_W and L_W do not depend on ε). Thus $p(\varepsilon)$ is interpreted as the probability that one particle with vanishing azimuthal quantum number will induce one disintegration ($\lambda^2/4\pi$ is, in fact, the cross section for such particles). This probability has a maximum for $\varepsilon = 0$, that is for the most favorable value of the energy, and p_0 can reach the unit value for $k = 1$ (see Eq. (51)). As pointed out by Majorana, these means that if the state ψ_0 has the same probability to emit a proton or an α -particle, and the energy of the incident α -particles takes its most favorable value, then all the incident particles with vanishing azimuthal quantum number will induce disintegration.

The analysis ends with the observation that when it is impossible to directly measure the cross section $S(\varepsilon)$ for particles with definite energy $E_0 + k + \varepsilon$, the measurable quantity of interest is only $\int S(\varepsilon)d\varepsilon$ that is given, according to Majorana, by the expression

$$\int S(\varepsilon)d\varepsilon = \frac{\lambda^2}{4\pi} \frac{\pi\hbar}{2T} p_0 = \frac{\hbar\lambda^2}{2T} \frac{k}{(k+1)^2}. \quad (67)$$

A complete theory for the artificial disintegration of nuclei induced by α -particles, which is fully accessible to experimental investigation, is thus finally provided.

5. CONCLUSIONS

In the historical reconstruction of the path towards the understanding of the structure of atomic nuclei, it is usually paid little attention to the experiments performed

on the artificial disintegration of nuclei by bombardment with α -particles. This is probably due to the (presumed) lack of a comprehensive theoretical interpretation of them and its consistent inclusion in the generally accepted framework of Quantum Mechanics applied to Nuclear Physics. In this respect, the original results from Chadwick and Rutherford in 1929, on the peculiarities of proton emission in the artificial disintegration of some nuclei, were puzzling when assuming the capture of the incident α -particle by the atomic nucleus. Only later, in 1930-31, with the fundamental contribution of Gamow, it was recognized that such disintegrations could take place even without the capture of the α -particles, with the assumption that the protons and α -particles contained in a nucleus are in definite energy levels. Thus, in general, the energy spectrum of the ejected protons resulted to be the superposition of a continuous spectrum and a line, as observed experimentally. Once realized this, the next step should have been the elaboration of a complete theory of such processes in the general framework of Quantum Mechanics. However, as a matter of fact, this was not even considered, as far as we know, by the main theorists who were working on that. Only recently we have realized that such an issue was effectively studied by Majorana in his personal researches, although he didn't publish anything on it. As shown in the previous pages, this was simply achieved by following the dynamics of a state resulting from the superposition of a discrete state with a continuous one, interacting between them through a term like in Eq. (2). The physical problem at hand was, then, approached by considering the simplest case with an unstable state of the system formed by a nucleus plus an α -particles, which spontaneously decays with the emission of an α -particle and a proton. As a result, Majorana succeeded in obtaining, among the other predictions, the explicit expression for the integrated cross section of the nuclear process, which was the direct measurable quantity of interest in the experiments. It should be noted that quasi-stationary states (the superposition of discrete and continuous states) were earlier introduced by Rice in a completely different context, that is the phenomenon of predissociation in molecules [17]. We do not know if Majorana was aware of the Rice's papers, but we have to observe that, disregarding the dissimilar theoretical elaborations, even the notations used by the two authors appear very different between them. Instead it is quite remarkable that another name is usually associated to the study of the quantum interference phenomenon between a discrete level and a continuum, i.e. the name of U. Fano. Indeed, the fundamental paper cited by almost everyone in this respect was that written by Fano at the beginning of 1935 [18], where he investigated the stationary states with configuration mixing under conditions of autoionization, when interpreting the strange looking shapes of spectral absorption lines of atoms in the continuum. The important result achieved by this author in an Atomic Physics framework was later related with the resonant scattering of a slow neutron in a nucleus, that is the so-called "shape resonances" found by Fermi [19]. The starting point of Fano, namely the writing of the perturbed eigenfunction of the system considered, as in the last equation on page 156 of Ref. [18], is surprisingly similar to what introduced by Majorana in his work of 1930. It is even remarkable to note the curious coincidence that Fano elaborated his theory just after few time when he moved in 1934-35 to Rome to work with Fermi [19]. Then it cannot be excluded a certain influence of Majorana, even indirectly (in his paper, Fano acknowledged the help given to him by Fermi), on this issue. In any case, it is certainly astonishing that a complete theory, with

application of the concept of the quasi-stationary states to a particular Nuclear Physics issue, was achieved by Majorana as early as in 1930, well before that the Fermi group decided to switch from Atomic Physics matters to those related to the structure of nuclei. Some other surprises we should expect from further studies of the unpublished papers of the Italian scientist.

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